Using the Definition of the Derivative to Evaluate Limits

(math guy going backwards again …)

Let's take a look at the following definition of the derivative. When we say "definition of the derivative", we aren't talking about ANY of the rules you learned. We are talking about the LIMIT definition (derivative = slope of tangent line = limit of secant lines).

Our particular one today:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Notice that both the left hand side (lhs) and the right hand side (rhs) are functions. In other words, we can evaluate the function at any particular value of x, by substituting that value in everywhere I see an x.

How do I know where the function is defined? It's a great question – but I'm not talking about that today ☺ I will say that the function is said to be differentiable at any point at which the derivative is defined ….

Let $f(x)=x^2$. Let's determine the derivative from the definition….

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}=\lim_{h\to 0}\frac{2xh+h^2}{h}=\lim_{h\to 0}\frac{h(2x+h)}{h}$$
$$=\lim_{h\to 0}2x+h=2x$$

Make sure you understand each step. Write the function with (x+h) substituted in for x; and then just with x – this is what you did with composition of functions. At the end of the first line, we factor out the x to make explicit what factors we are going to cancel (the two 'h's).

And notice how the limit is calculated … we multiply things out until we can cancel the 'h' in the denominator with something in the numerator that is in EACH term.

We're going to zone in on that second limit: $\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$. What if I replace 'x' with a number – like 4?

Then the limit becomes: $\lim_{h\to 0}\frac{(4+h)^2-4^2}{h}$. The method I used to find the limit (with the x above) can also be used to calculate this limit. BUT there is a simpler way … If I recognize that last limit as the derivative of $f(x)=x^2$, evaluated at x=4 then I can just say: $\lim_{h\to 0}\frac{(4+h)^2-4^2}{h}=f'(4)=2x\big|_{x=4}=2(4)=8$. And my accuracy rate goes up as well.

Write the limit definition of the following:

1) $f(x)=x^3.f'(-2)$. The answer I want is: $f'(-2)=\lim_{h\to 0}\frac{(-2+h)^3-(-2)^3}{h}=3x^2\big|_{x=-2}$

2) $f(x)=\sin(x).f'(\pi)$

3) $f(x)=\sqrt{x^2+9}.f'(3)$

The real value is recognizing the limit the other way. Here are some limits you can practice visualizing a different way:

A1) $\lim\limits_{h\to 0}\dfrac{\tan(x+h)-\tan(x)}{h}$

A2) $\lim\limits_{h\to 0}\dfrac{(x+h)^4-(x)^4}{h}$

A3) $\lim\limits_{h\to 0}\dfrac{2(x+h)^2+3(x+h)-2x^2-3h}{h}$. A tricky one – rewrite as: $\lim\limits_{h\to 0}\dfrac{2(x+h)^2+3(x+h)-(2x^2+3h)}{h}$

A4) $\lim\limits_{h\to 0}\dfrac{\sin(2(x+h))-\sin(2x)}{h}$

A5) $\lim\limits_{h\to 0}\dfrac{\cos(x+h)^2-\cos(x^2)}{h}$

The following examples use a different definition of the limit, and are the values of a derivative at a particular x-value.

B1) $\lim\limits_{x\to 5}\dfrac{(x^2-5^2)}{x-5}$. Here the function is $f(x)=x^2$. The particular x-value is 5.

B2) $\lim\limits_{x\to 5}\dfrac{(x^2-25)}{x-5}$

B3) $\lim\limits_{x\to(\pi/3)}\dfrac{(\sin(x)-\sin\left(\frac{\pi}{3}\right))}{x-\left(\frac{\pi}{3}\right)}$

B4) $\lim\limits_{x\to(\pi/3)}\dfrac{(\sin(x)-\left(\frac{\sqrt{3}}{2}\right))}{x-\left(\frac{\pi}{3}\right)}$. A trickier version of the above ….

B5) $\lim\limits_{x\to(\pi/4)}\dfrac{(\cos(x)-\left(\frac{\sqrt{2}}{2}\right))}{x-\left(\frac{\pi}{4}\right)}$.